GENERAL SOLUTIONS TO EQUATION $axb^* - bx^*a^* = c$ IN RINGS WITH INVOLUTION

CHAO YOU, CHANGHUI WANG, AND YICHENG JIANG

This paper is dedicated to Prof. Lixin Xuan.

ABSTRACT. In [Q. Xu et al., The solutions to some operator equations, Linear Algebra Appl.(2008), doi:10.1016/j.laa.2008.05.034], Xu et al. provided the necessary and sufficient conditions for the existence of a solution to the equation $AXB^* - BX^*A^* = C$ in the general setting of the adjointable operators between Hilbert C^*-modules. Based on the generalized inverses, they also obtained the general expression of the solution in the solvable case. In this paper, we generalize their work in the more general setting of ring $\mathscr{R}$ with involution $*$ and reobtain results for rectangular matrices and operators between Hilbert C^*-modules by embedding the "rectangles" into rings of square matrices or rings of operators acting on the same space.

INTRODUCTION

Let $R(A)$ be the range of a matrix or an operator. The equation $AXB^* - BX^*A^* = C$ was studied by Yuan [1] and Xu et al. [2], for finite matrices and adjointable operators between Hilbert C^*-modules respectively, under the condition that $R(B) \subseteq R(A)$. When A equals an identity matrix or identity operator, this equation reduces to $XB^* - BX^* = C$, which was studied by Braden [3] for finite matrices, and Djordjević [4] for the Hilbert space operators.

In this paper, we turn our attention to the equations $axb^* - bx^*a^* = c$ where a, b, c and x are elements of a ring $\mathscr{R}$ with involution. This point of view emphasizes the purely algebraic nature of the problem without regard to specific properties of matrices or bounded linear operators, and reveals the intrinsic simplicity of the solutions. Thus the equations are studied in a greater generality and in a transparent environment.

A novel feature of our paper is that the results for finite rectangular matrices and adjointable operators between Hilbert C^*-modules are derived from theorems for rings with involution using the method of embedding decribed in Section 3.

Having established preliminary settings in Section 1, we study the equation $AXB^* - BX^*A^* = C$ in Section 2 in the setting of rings with involution, giving necessary and sufficient conditions for the existence, and the general form of these solutions based on Moore-Penrose inverses.

Section 3 is concerned with the extensions of the preceding results to finite rectangular matrices with entries in a ring with involution, and to adjointable operators

between Hilbert C^*-modules. This is achieved by embedding the rectangular matrices as blocks into the ring of square matrices of the same order, and by embedding rectangular operators via operator matrices into the ring of operators acting on the same space. This shows that our work is genuinely a generalization of previous results, and a progress in the theory of equations in this type.

1. Preliminaries

Throughout this paper, ring $\mathscr{R}$ will always mean an involution ring with a unit $1 \neq 0$ such that 2 is invertible in $\mathscr{R}$. An involution $*$ is a unary operation $a \mapsto a^*$ on $\mathscr{R}$ preserved by the addition $((a+b)^* = a^* + b^*)$, reversed by the multiplication $((ab)^* = b^*a^*)$ and satisfying $(a^*)^* = a$ and $1^* = 1$.

If $\mathscr{R}$ is a ring with involution and $a \in \mathscr{R}$, we say that $b \in \mathscr{R}$ is a *Moore-Penrose inverse* of a, or *MP-inverse* for short, if it satisfies the *Penrose equations* [6]:

$$(1.1) \quad aba = a, \quad bab = b, \quad (ab)^* = ab, \quad (ba)^* = ba.$$

Proposition 1.1. *Let $\mathscr{R}$ be a ring with involution and $a \in \mathscr{R}$. If $b_1, b_2 \in \mathscr{R}$ are both MP-inverses of a, then $b_1 = b_2$.*

Proof. $b_1 a = b_1 a b_2 a = a^* b_1^* a^* b_2^* = a^* b_2^* = b_2 a$. Similarly, $ab_1 = ab_2$. Hence $b_1 = b_1 a b_1 = b_2 a b_1 = b_2 a b_2 = b_2$. $\qquad\square$

From Proposition 1.1, we know that the MP-inverse of a is unique if it exists, and is denoted by $a^\dagger$. If the MP-inverse $a^\dagger$ of a exists, we say that a is *Moore-Penrose invertible*, or *MP-invertible* for short.

Proposition 1.2. 2^{-1}, *the inverse of 2 in $\mathscr{R}$, commutes with every element in $\mathscr{R}$.*

Proof. For any $r \in \mathscr{R}$, $2r = (1+1)r = r + r = r(1+1) = r2$. Multiplying 2^{-1} from both left and right sides of the equality above, we get $r2^{-1} = 2^{-1}r$ as desired. $\quad\square$

Since, moreover, it can be easily checked that $2^{-1} + 2^{-1} = 1$ in $\mathscr{R}$, we can see that 2^{-1} functions just as number "$\frac{1}{2}$" in the calculations within $\mathscr{R}$. Hence, without any confusion, we will denote 2^{-1} by $\frac{1}{2}$ throughout this paper.

More notations are needed. In this paper, E_a, F_a, $H^{(+,*)}(a)$ and $H^{(-,*)}(a)$ are reversed to denote $1 - aa^\dagger$, $1 - a^\dagger a$, $a + a^*$ and $a - a^*$, respectively. Please keep in mind that E_a, F_a are projections, and $(H^{(+,*)}(a))^* = H^{(+,*)}(a)$, $(H^{(-,*)}(a))^* = -H^{(-,*)}(a)$, which will be very useful in the calculations later.

2. General solutions to the equation $axb^* - bx^*a^* = c$ in the setting of rings with involution

In this section, we will study the general solutions to Eq.(2.8) below in the general setting of rings with involution.

Lemma 2.1. *If a, b are MP-invertible elements in $\mathscr{R}$ such that $aa^\dagger b = b$ and $(a^\dagger bb^\dagger a)^* = a^\dagger bb^\dagger a$, then $d = E_b a$ is also MP-invertible with the unique MP-inverse $d^\dagger = a^\dagger E_b$.*

Proof. Check the conditions of (1.1):

$$(E_b a)(a^\dagger E_b)(E_b a) = E_b aa^\dagger E_b a = E_b aa^\dagger (1 - bb^\dagger) a = E_b(aa^\dagger a - aa^\dagger bb^\dagger a)$$

$$= E_b(a - bb^\dagger a) = E_b(1 - bb^\dagger)a = E_b E_b a = E_b a.$$

Since $aa^\dagger b = b$, then $aa^\dagger bb^\dagger = bb^\dagger$, and $bb^\dagger aa^\dagger = bb^\dagger$.

$$(a^\dagger E_b)(E_b a)(a^\dagger E_b) = a^\dagger E_b aa^\dagger E_b = a^\dagger(1 - bb^\dagger)aa^\dagger E_b = (a^\dagger aa^\dagger - a^\dagger bb^\dagger aa^\dagger)E_b$$
$$= (a^\dagger - a^\dagger bb^\dagger)E_b = a^\dagger(1 - bb^\dagger)E_b = a^\dagger E_b E_b = a^\dagger E_b.$$

$$((E_b a)(a^\dagger E_b))^* = (E_b(aa^\dagger)E_b)^* = E_b(aa^\dagger)E_b = (E_b a)(a^\dagger E_b).$$
$$((a^\dagger E_b)(E_b a))^* = (a^\dagger E_b a)^* = (a^\dagger a - a^\dagger bb^\dagger a)^* = a^\dagger a - a^\dagger bb^\dagger a = (a^\dagger E_b)(E_b a).$$

Thus d is MP-invertible and $d^\dagger = a^\dagger E_b$. $\qquad\square$

Theorem 2.2. *Let a, b be MP-invertible elements in $\mathscr{R}$ such that $aa^\dagger b = b$ and $(a^\dagger bb^\dagger a)^* = a^\dagger bb^\dagger a$, and $d = E_b a$. Then the general solution $x \in \mathscr{R}$ to the equation*

$$(2.1) \quad axb^* - bx^*a^* = 0$$

is of the form

$$(2.2) \quad x = v - \frac{1}{2}a^\dagger avb^\dagger b + \frac{1}{2}a^\dagger bv^*a^*(b^\dagger)^* - \frac{1}{2}a^\dagger bv^*(b^\dagger ad^\dagger a)^* - \frac{1}{2}d^\dagger avb^\dagger b,$$

where $v \in \mathscr{R}$ is arbitrary.

Proof. Since $d = E_b a$ by definition, we have $E_b d = d$, hence $E_b dd^\dagger = dd^\dagger$. Taking $*$-operation, we get $dd^\dagger = dd^\dagger E_b$. It follows that

$$(2.3) \quad d^\dagger b = d^\dagger(dd^\dagger b) = d^\dagger(dd^\dagger E_b b) = 0, \quad b^* dd^\dagger = (dd^\dagger b)^* = 0,$$

$$(2.4) \quad dd^\dagger a = dd^\dagger(bb^\dagger + E_b)a = dd^\dagger(E_b a) = dd^\dagger d = d,$$

$$(2.5) \quad d^\dagger a = d^\dagger(dd^\dagger a) = d^\dagger d.$$

For any $v \in \mathscr{R}$, let

$$(2.6) \quad \Phi(v) = v - \frac{1}{2}a^\dagger avb^\dagger b + \frac{1}{2}a^\dagger bv^*a^*(b^\dagger)^* - \frac{1}{2}a^\dagger bv^*(b^\dagger ad^\dagger a)^* - \frac{1}{2}d^\dagger avb^\dagger b.$$

In view of (2.5) and the definition of d, we have

$$a\Phi(v)b^* = avb^* - \frac{1}{2}avb^* + \frac{1}{2}bv^*a^*bb^\dagger - \frac{1}{2}bv^*(bb^\dagger ad^\dagger a)^* - \frac{1}{2}ad^\dagger avb^*$$

$$= \frac{1}{2}avb^* + \frac{1}{2}bv^*a^*bb^\dagger - \frac{1}{2}bv^*((1 - E_b)ad^\dagger d)^* - \frac{1}{2}ad^\dagger dvb^*$$

$$= \frac{1}{2}avb^* + \frac{1}{2}bv^*a^*bb^\dagger - \frac{1}{2}bv^*d^\dagger da^* + \frac{1}{2}bv^*d^* - \frac{1}{2}ad^\dagger dvb^*$$

$$= \frac{1}{2}avb^* + \frac{1}{2}bv^*a^*bb^\dagger + \frac{1}{2}bv^*a^*E_b - H^{(+,*)}(\frac{1}{2}bv^*d^\dagger da^*)$$

$$= H^{(+,*)}(\frac{1}{2}avb^*) - H^{(+,*)}(\frac{1}{2}bv^*d^\dagger da^*).$$

It follows that $\Phi(v)$ is a solution to Eq.(2.1).

On the other hand, given any solution $x \in \mathscr{R}$ to Eq. (2.1), let $v = x$. Then since $d^\dagger b = 0$, we have

$$\Phi(x) = x - \frac{1}{2}a^\dagger axb^\dagger b + \frac{1}{2}a^\dagger(bx^*a^*)(b^\dagger)^* - \frac{1}{2}a^\dagger(bx^*a^*)(b^\dagger ad^\dagger)^* - \frac{1}{2}d^\dagger axb^\dagger b$$

$$= x - \frac{1}{2}a^\dagger axb^\dagger b + \frac{1}{2}a^\dagger axb^*(b^\dagger)^* - \frac{1}{2}a^\dagger axb^*(b^\dagger ad^\dagger)^* - \frac{1}{2}d^\dagger axb^*(b^\dagger)^*$$

$$= x - \frac{1}{2}a^\dagger ax(b^\dagger ad^\dagger b)^* - \frac{1}{2}d^\dagger bx^*a^*(b^\dagger)^* = x.$$

We have proved that the general solution to Eq.(2.1) has a form $\Phi(v)$ for some $v \in \mathscr{R}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 2.3. *Let a, b be MP-invertible elements in $\mathscr{R}$ such that $aa^\dagger b = b$ and $(a^\dagger bb^\dagger a)^* = a^\dagger bb^\dagger a$, and $d = E_b a$. Then*

$$(2.7) \quad x_0 = \frac{1}{2}a^\dagger c(b^\dagger)^* - \frac{1}{2}a^\dagger bb^\dagger c(b^\dagger ad^\dagger)^* + \frac{1}{2}d^\dagger c(b^\dagger)^*$$

is a solution to the equation

$$(2.8) \quad axb^* - bx^*a^* = c$$

if and only if

$$(2.9) \quad c^* = -c \quad and \quad H^{(-,*)}((aa^\dagger + dd^\dagger)cbb^\dagger) = 2c.$$

Proof. If x_0 is a solution to equation (2.8), then obviously $c^* = -c$ and since $d = E_b a$, in view of (2.3) to (2.5) we have

$$aa^\dagger cbb^\dagger = aa^\dagger(ax_0b^* - bx_0^*a^*)bb^\dagger = ax_0b^* - bx_0^*a^*bb^\dagger,$$
$$bb^\dagger caa^\dagger = (aa^\dagger(-c)bb^\dagger)^* = -bx_0^*a^* + bb^\dagger ax_0b^*,$$
$$bb^\dagger cdd^\dagger = bb^\dagger(ax_0b^* - bx_0^*a^*)dd^\dagger = -bx_0^*a^*dd^\dagger = -bx_0^*d^* = -bx_0^*a^*E_b,$$
$$bb^\dagger cdd^\dagger + dd^\dagger cbb^\dagger = -bx_0^*a^*E_b + E_bax_0b^*.$$

Therefore,

$$H^{(-,*)}((aa^\dagger + dd^\dagger)cbb^\dagger)$$
$$=aa^\dagger cbb^\dagger + bb^\dagger caa^\dagger + dd^\dagger cbb^\dagger + bb^\dagger cdd^\dagger$$
$$=ax_0b^* - bx_0^*a^* + (bb^\dagger + E_b)ax_0b^* - bx_0^*a^*(bb^\dagger + E_b)$$
$$=2(ax_0b^* - bx_0^*a^*) = 2c.$$

Conversely, suppose that (2.9) is satisfied. Let x_0 be defined by (2.7). Then as $c^* = -c$ and $d = E_b a$, we have

$$ax_0b^* = \frac{1}{2}aa^\dagger cbb^\dagger - \frac{1}{2}bb^\dagger c(bb^\dagger ad^\dagger)^* + \frac{1}{2}ad^\dagger cbb^\dagger$$
$$= \frac{1}{2}aa^\dagger cbb^\dagger - \frac{1}{2}bb^\dagger c((1 - E_b)ad^\dagger)^* + \frac{1}{2}ad^\dagger cbb^\dagger$$
$$= \frac{1}{2}aa^\dagger cbb^\dagger - \frac{1}{2}bb^\dagger c(ad^\dagger)^* + \frac{1}{2}bb^\dagger c(dd^\dagger)^* + \frac{1}{2}ad^\dagger cbb^\dagger$$
$$= \frac{1}{2}H^{(+,*)}(ad^\dagger cbb^\dagger) + \frac{1}{2}aa^\dagger cbb^\dagger + \frac{1}{2}bb^\dagger cdd^\dagger,$$

So

$$H^{(-,*)}(ax_0b^*) = ax_0b^* - bx_0^*a^*$$
$$= \frac{1}{2}aa^\dagger cbb^\dagger + \frac{1}{2}bb^\dagger cdd^\dagger - (\frac{1}{2}aa^\dagger cbb^\dagger)^* - (\frac{1}{2}bb^\dagger cdd^\dagger)^*$$
$$= \frac{1}{2}(aa^\dagger cbb^\dagger + bb^\dagger cdd^\dagger + bb^\dagger caa^\dagger + dd^\dagger cbb^\dagger)$$
$$= \frac{1}{2}H^{(-,*)}((aa^\dagger + dd^\dagger)cbb^\dagger) = c,$$

which means that x_0 is a solution to Eq.(2.8). $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now, we arrive at the most important result of this paper.

Theorem 2.4. *Let a, b be MP-invertible elements in $\mathscr{R}$ such that $aa^\dagger b = b$ and $(a^\dagger bb^\dagger a)^* = a^\dagger bb^\dagger a$, and $d = E_b a$. Then Eq.(2.8) has a solution if and only if (2.9) holds. In which case, the general solution x to Eq.(2.8) is of the form $x = x_0 + \Phi(v)$, where $v \in \mathscr{R}$ is arbitrary, and $x_0, \Phi(v)$ are defined by (2.7) and (2.6), respectively.*

In fact, if applying the procedure above to the equation

$$(2.10) \quad axb^* + bx^*a^* = c,$$

we can get a result similar to Theorem 2.4, which we will give in the following theorem, leaving the proof to the read as an exercise.

Theorem 2.5. *Let a, b be MP-invertible elements in $\mathscr{R}$ such that $aa^\dagger b = b$ and $(a^\dagger bb^\dagger a)^* = a^\dagger bb^\dagger a$, and $d = E_b a$. Then Eq.(2.10) has a solution if and only if $c^* = c$ and $H^{(+,*)}((aa^\dagger + dd^\dagger)cbb^\dagger) = 2c$. In which case, the general solution x to Eq.(2.10) is of the form $x = x_0' + \Phi'(v)$, where $v \in \mathscr{R}$ is arbitrary, and x_0 and $\Phi(v)$ are defined as the following, respectively:*

$$(2.11) \quad x_0' = \frac{1}{2}a^\dagger c(b^\dagger)^* - \frac{1}{2}a^\dagger bb^\dagger c(b^\dagger ad^\dagger)^* + \frac{1}{2}d^\dagger c(b^\dagger)^*$$

$$(2.12) \quad \Phi'(v) = v - \frac{1}{2}a^\dagger avb^\dagger b - \frac{1}{2}a^\dagger bv^*a^*(b^\dagger)^* + \frac{1}{2}a^\dagger bv^*(b^\dagger ad^\dagger a)^* - \frac{1}{2}d^\dagger avb^\dagger b.$$

If we replace a, b, and c by 1, a and b, respectively. From Theorem 2.5, we get the following corollary.

Corollary 2.6. *Let $a \in \mathscr{R}$ be MP-invertible and $b \in \mathscr{R}$. Then the equation*

$$(2.13) \quad xa^* + ax^* = b$$

has a solution if and only if

$$(2.14) \quad b^* = b \quad and \quad E_a b E_a = 0.$$

In which case, the general solution x to Eq.(2.13) is of the form

$$(2.15) \quad x = \frac{1}{2}(1 + E_a)(b(a^\dagger)^* - va^\dagger a) + v - \frac{1}{2}av^*(a^\dagger)^*,$$

where $v \in \mathscr{R}$ is arbitrary.

Similarly, for the equation

$$(2.16) \quad a^*x + x^*a = b,$$

we also have the following corollary.

Corollary 2.7. *Let $a \in \mathscr{R}$ be MP-invertible and $b \in \mathscr{R}$. Then Eq.(2.16) has a solution if and only if*

$$(2.17) \quad b^* = b \quad and \quad F_a b F_a = 0.$$

In which case, the general solution x to Eq.(2.16) is of the form

$$(2.18) \quad x = \frac{1}{2}((a^\dagger)^*b - aa^\dagger w)(1 + F_a) + w - \frac{1}{2}(a^\dagger)^*w^*a,$$

where $w \in \mathscr{R}$ is arbitrary.

3. The embedding: from rings to rectangular matrices and adjointable operators between Hilbert C^*-modules

In this section, we will use the method described in [5] to extend the results for ring $\mathscr{R}$ with involution to the rectangular matrices over $\mathscr{R}$ and adjointable operators between Hilbert C^*-modules.

The results of the preceding section apply to square matrices of the same order n over a ring $\mathscr{R}$ as these form a ring $\mathscr{R}^{n \times n}$ under the usual matrix operations and with the involution defined as involute transpose. Suppose that $A \in \mathscr{R}^{m \times n}$, $B \in \mathscr{R}^{p \times m}$ and $C \in \mathscr{R}^{m \times m}$, consider the equation

(3.1) $\quad AXB^* - BX^*A^* = C \quad$ for $\quad X \in \mathscr{R}^{n \times p}$.

The embedding of Eq.(3.1) to a ring is achieved by defining a, b and c in the ring $\mathscr{R}^{k \times k}$, where $k = m + n + p$, by

(3.2) $\quad a = \begin{bmatrix} 0 & A & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad$ and $\quad c = \begin{bmatrix} C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Now consider the equation

(3.3) $\quad axb^* - bx^*a^* = c, \quad$ for $\quad x \in \mathscr{R}^{k \times k}$,

which can also be expressed in the following detailed matrix form,

(3.4)
$$\begin{bmatrix} 0 & A & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ B^* & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_{11}^* & X_{21}^* & X_{31}^* \\ X_{12}^* & X_{22}^* & X_{32}^* \\ X_{13}^* & X_{23}^* & X_{33}^* \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ A^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Straightforward calculation shows that Eq.(3.4) is equal to the following equation

(3.5) $\quad \begin{bmatrix} AX_{23}B^* - BX_{23}^*A^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

So we have the following lemma.

Lemma 3.1. *Let $A \in \mathscr{R}^{m \times n}$, $B \in \mathscr{R}^{p \times m}$ and $C \in \mathscr{R}^{m \times m}$, let a, b and c be defined by (3.2), and let $k = m + n + p$. Then Eq.(3.1) has a solution $X \in \mathscr{R}^{n \times p}$ if and only if Eq.(3.3) has a solution $x \in \mathscr{R}^{k \times k}$ with $X_{23} = X$. In this case there is a one-to-one correspondence between the solution X of Eq.(3.1) and the solution x of Eq.(3.3) of the form*

(3.6) $\quad x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & X \\ 0 & 0 & 0 \end{bmatrix}$.

Note that, if A and B have MP-inverses $A^\dagger \in \mathscr{R}^{n \times m}$ and $B^\dagger \in \mathscr{R}^{m \times p}$, respectively, then a and b are also MP-invertible with

(3.7) $\quad a^\dagger = \begin{bmatrix} 0 & 0 & 0 \\ A^\dagger & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad b^\dagger = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ B^\dagger & 0 & 0 \end{bmatrix}$.

Then it is a work of direct calculation to get the following theorem, which was obtained by [1] in the case of real matrices.

Theorem 3.2. *Let $\mathcal{R}$ be a ring with involution, let $A \in \mathcal{R}^{m \times n}$, $B \in \mathcal{R}^{p \times m}$ be MP-invertible and $AA^{\dagger}B = B$, $(A^{\dagger}BB^{\dagger}A)^* = A^{\dagger}BB^{\dagger}A$, $D = E_B A$. Then*

$$(3.8) \quad X_0 = \frac{1}{2}A^{\dagger}C(B^{\dagger})^* - \frac{1}{2}A^{\dagger}BB^{\dagger}C(B^{\dagger}AD^{\dagger})^* + \frac{1}{2}D^{\dagger}C(B^{\dagger})^*$$

is a solution to Eq.(3.1) if and only if

$$(3.9) \quad C^* = -C \quad and \quad H^{(-,*)}((AA^{\dagger} + DD^{\dagger})CBB^{\dagger}) = 2C.$$

In which case, the general solution X to Eq.(3.1) is of the form $X = X_0 + \Phi(V)$, where $V \in \mathcal{R}^{n \times p}$ is arbitrary, and $\Phi(V)$ is defined by (2.6).

Now we turn to the case of adjointable operator between Hilbert C^*-modules. Let H_1, H_2 and H_3 be Hilbert C^*-modules, let $A \in \mathcal{L}(H_3, H_2)$, $B \in \mathcal{L}(H_1, H_2)$ and $C \in \mathcal{L}(H_2)$ be adjointable operators. The solvability of the equation

$$(3.10) \quad AXB^* - BX^*A^* = C \quad for \quad X \in \mathcal{L}(H_1, H_3)$$

was studied in [2]. Now we let $H = H_1 \oplus H_2 \oplus H_3$,

$$(3.11) \quad \widetilde{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & A \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{L}(H),$$

$$(3.12) \quad \widetilde{B} = \begin{bmatrix} 0 & 0 & 0 \\ B & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{L}(H), \quad and$$

$$(3.13) \quad \widetilde{C} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{L}(H).$$

Consider the equation

$$(3.14) \quad \widetilde{A}\widetilde{X}\widetilde{B}^* - \widetilde{B}\widetilde{X}^*\widetilde{A}^* = \widetilde{C}, \quad for \widetilde{X} \in \mathcal{L}(H, H).$$

Similar to Lemma 3.1, we have

Lemma 3.3. *Let H_1, H_2 and H_3 be Hilbert C^*-modules, $A \in \mathcal{L}(H_3, H_2)$, $B \in \mathcal{L}(H_1, H_2)$ and $C \in \mathcal{L}(H_2)$ be adjointable operators. Then Eq.(3.10) has a solution $X \in \mathcal{L}(H_1, H_3)$ if and only if Eq.(3.14) has a solution $\widetilde{X} \in \mathcal{L}(H)$ with $\widetilde{X}_{31} = X \in \mathcal{L}(H_1, H_3)$. In this case there is a one-to-one correspondence between the solution $\widetilde{X}$ of Eq.(3.14) and the solution X of Eq.(3.10) of the form*

$$(3.15) \quad \widetilde{X} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ X & 0 & 0 \end{bmatrix}.$$

Note that in Hilbert C^*-modules, if $A \in \mathcal{L}(H_3, H_2)$ is MP-invertible, then $R(B) \subseteq R(A)$ is equivalent to $AA^{\dagger}B = B$, and that for any operator A, A is MP-invertible if and only if $R(A)$ is closed. From Theorem 2.4, we reobtain the following theorem, which was first given in [2].

Theorem 3.4. *Let H_1, H_2 and H_3 be Hilbert C^*-modules, $A \in \mathscr{L}(H_3, H_2)$, $B \in \mathscr{L}(H_1, H_2)$ and $C \in \mathscr{L}(H_2)$ be adjointable operators such that $R(B) \subseteq R(A)$ and $D = E_B A$ such that $R(D)$ is also closed. Then Eq.(3.10) has a solution if and only if*

$$(3.16) \quad C^* = -C \quad and \quad H^{(-,*)}((AA^\dagger + DD^\dagger)CBB^\dagger) = 2C.$$

In which case, the general solution X to Eq.(3.10) is of the form

$$X = X_0 + V - \frac{1}{2}A^\dagger A V B^\dagger B + \frac{1}{2}A^\dagger B V^* A^*(B^\dagger)^*$$
$$- \frac{1}{2}A^\dagger B V^*(B^\dagger A D^\dagger A)^* - \frac{1}{2}D^\dagger A V B^\dagger B,$$

where $V \in \mathscr{L}(H_1, H_3)$ is arbitrary, and X_0 is a particular solution to Eq.(3.10) defined by

$$(3.17) \quad X_0 = \frac{1}{2}A^\dagger C(B^\dagger)^* - \frac{1}{2}A^\dagger B B^\dagger C(B^\dagger A D^\dagger)^* + \frac{1}{2}D^\dagger C(B^\dagger)^*.$$

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DEPARTMENT OF MATHEMATICS & THE ACADEMY OF FUNDAMENTAL AND INTERDISCIPLINARY SCIENCE, HARBIN INSTITUTE OF TECHNOLOGY, HARBIN 150001, HEILONGJIANG, PEOPLE'S REPUBLIC OF CHINA
E-mail address: `hityou1982@gmail.com`

DEPARTMENT OF MATHEMATICS, HARBIN INSTITUTE OF TECHNOLOGY, HARBIN 150001, HEILONGJIANG, PEOPLE'S REPUBLIC OF CHINA
E-mail address: `shuzihero@gmail.com`

DEPARTMENT OF MATHEMATICS, QIQIHAR UNIVERSITY, QIQIHAR, 161006, HEILONGJIANG, PEOPLE'S REPUBLIC OF CHINA
E-mail address: `hitjyc@gmail.com`